STRUCTURING OF LARGE AND HETEROGENEOUS TEXTURE DATABASES

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ABSTRACT

Processing large databases is intricate for databases involving several types of textures. In particular, for content-based image retrieval, a query has to be compared with all the samples pertaining to the database in order to identify its content/class and this is time consuming. Furthermore, modeling of a large database is a difficult task for databases involving several types of textures since accurate models for certain textures are not guaranteed to be very relevant for other types of textures and *vice-versa*. In order to save computational time and increase performance in processing of large texture databases, the present paper proposes structuring texture databases by using stochasticity metaclasses.

Index Terms— Texture ; Stochasticity ; Regularity ; Stationary Wavelet Transform ; Similarity Measurements ; Edgeworth expansion ; Generalized Gaussian Distribution ; Pareto Distribution ; Weibull distribution ; Content-Based Image Retrieval.

1. INTRODUCTION

Wavelet transforms have proven useful in texture characterization *via* stochastic parametric modeling. This is due to the suitable statistical properties characterizing the wavelet coefficients issued from the decomposition of stochastic processes: dependency reduction occurs among the wavelet coefficients (see [1], [2], among others) and makes relevant, the use of statistical modeling under independence assumption on the wavelet subbands (see the overview given in [3]).

The present paper addresses structuring of texture databases with respect to stochasticity measurements in the wavelet domain. Structuring a database consists in identifying texture classes (metaclass, structure) that are likely to be accurately described by stochastic models, due to their randomness-like behavior in the wavelet domain. In this respect, searching a *stochastic texture* will be restricted to its associated metaclass and is expected to yield good retrieval performance. Experimental results concerning Content Based Image Retrieval, CBIR, show that stochasticity pre-consideration is relevant for the post-processings of large texture databases containing both deterministic and stochastic type textures.

Specifically, almost all deterministic <u>textures</u> are those textures that are geometrically regular in the sense of [4] and, consequently, we will use the terminology of *regular* textures to denote textures that are far from being *stochastic* in the wavelet domain.

2. METHOD DESCRIPTION

Let $\mathcal{T}$ be a texture database. We consider the problem of structuring the elements of $\mathcal{T}$ by using low-level texture features. From the literature, low-level features that have proven relevancy in texture analysis are coarseness, roughness, regularity, directionality and contrast.

The effect of directionality is fixed in the approach proposed hereafter in the sense that similarity measurements between features concern oriented and multi-scale wavelet subbands. In addition, contrast calibration is addressed by simply imposing the same mean value for all the subimages of a given texture class.

Coarseness and roughness are intrinsic to stochastic-type textures and can be captured through stochastic modeling. It is worth noticing that in texture images, many non-stochastic (or deterministic) patterns consist in smooth objects, while smoothness relates to regularity. Thus, when stochasticity holds true, regularity trivially fails. In this respect, we consider structuring the database in terms of metaclasses depending on the stochasticity degree. For the sake of simplifying the presentation, we will consider the splitting of database $\mathcal{T}$ in two metaclasses: stochastic versus regular textures.

Let x be a dataset having N-samples that are realization of some independent and identically distributed, *iid*, random variables with probability distribution function F. The Kolmogorov stochasticity index of x is [5]:

$$\kappa(x, F) = \sup_t |F_x(t) - F(t)|, \tag{1}$$

where F_x is the empirical cumulative distribution function of x. In testing random generators, admissible indices consist in values of $\kappa(x, F)$ that are almost certain, provided that the sample size N tends to infinity. The Kolmogorov distribution, derived from the asymptotic of the random variable $\sqrt{N} \times \kappa(x, F)$ is commonly used to fix these admissible indices. From the Kolmogorov distribution, the admissible indices can be chosen as those κ satisfying

$$\frac{0.3}{\sqrt{N}} \leqslant \kappa(x, F) \leqslant \frac{2.4}{\sqrt{N}}.$$

For a natural image (quantified pixel values with fixed dynamical range and finite sample size), $\kappa(x, F)$ is not expected to decrease significantly when the sample size increases. In order to assess the intrinsic stochasticity of the texture, we derive a bound on $\kappa(x, F)$ from a mean squared error consideration in the problem of fitting $F_x(t)$ by $F(t)$. This is performed by considering $\kappa(x, F)$ as an estimation (ℓ_∞) error and the admissible indices are those satisfying $\kappa\left(c_{j,n}, F\right) < \eta_0$, where η_0 is chosen so as to guarantee an appropriate PSNR. This leads, if we consider PSNRs greater than 35 dBs (good quality in image denoising and compression problems), to $\eta_0 = \sqrt{10^{-3.5}}$.

The second constraint imposed by the measure $\kappa(x, F)$ is the *iid* nature of the data. In order to approximately attain this condition, measurements are performed in the wavelet domain. In-

deed, wavelet based transforms have appreciable statistical properties such as stationarization, decorrelation and higher order dependency reduction for many random processes (see [1], [2], among others). These properties are obtained with respect to some key parameters that are: the shape of the polyspectra of the input random process, the wavelet order and the wavelet decomposition level (see the above references for more details). Furthermore, from these statistical properties, we can:

(1) restrict F to pertain to the class of distributions having probability density functions with exponential decay, since mainly distributions among this class are relevant for modeling the wavelet subbands.

(2) select the Symlet wavelet of order 8 as a relevant wavelet for the above statistical properties to be substantial.

The sequence of wavelet subband stochasticity measurements will then describes the stochasticity behavior of the input texture. Regular textures are such that there exists at least one detail wavelet subband with large stochasticity index.

It is worth mentioning that full wavelet packets yield better statistical representations in terms of the *iid* condition. In addition, shift-invariance is often desirable in signal and image representations when analysis is concerned. In particular, this shift-invariance makes the Stationary Wavelet Transform (SWT) and its wavelet packet version more relevant in texture analysis. However, due to the computational complexity of the stationary wavelet packets, we use the SWT in the following. The sole consequence of this alternative is that the class of stochastic textures is slightly reduced, which has no consequence on the method.

Figure 1 provides an illustration of some VisTeX[1] texture database structuring obtained in terms of stochastic versus regular textures. The detail subbands are considered in these experiments. Stochasticity is measured with respect to a dictionary composed of continuous cumulative distribution functions, among which the Generalized Gaussian and the Weibull distributions play an important role. The experimental results presented in Section 4 show that such a structuring is useful in CBIR. Before presenting these results, the following Section fixes the statistical tools used for the description of texture features.

3. PARAMETRIC MODELING AND SIMILARITY MEASUREMENTS

In this section, we address the SWT subband modeling and similarity measurements. It is worth highlighting that both stochastic and regular will be concerned by parametric modeling by using probability distribution functions. This issue is natural for stochastic textures. For regular textures, this issue is motivated by the following fact: the parametric models under consideration act as bests approximations of the sparse sequences of random variables representing the coefficients of these textures. Thus, these parametric models are expected to yield reasonable performance for regular images in CBIR, while the parametric modeling is expected to yield high CBIR performance for the class of stochastic textures.

3.1. Parametric modeling of the SWT subband coefficients

The SWT approximation coefficients have specific behavior because these coefficients are associated with a scaling function having no-vanishing moment. This is why approximation coefficients

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Stochastic textures from SWT based stochasticity

"Fabric.07"   "Fabric.04"   "Fabric.11"

Regular textures from SWT based stochasticity
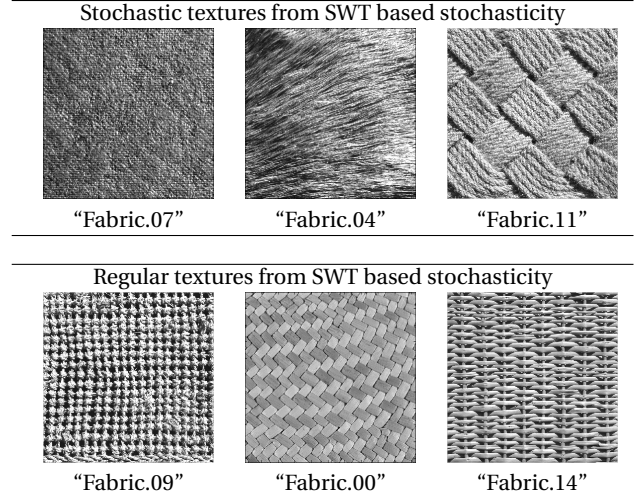
"Fabric.09"   "Fabric.00"   "Fabric.14"

Fig. 1. Database structuring in terms of stochastic versus regular textures. Stochasticity is measured in the SWT domain and concerns detail subbands, up to decomposition level $J = 4$. Texture images considered are the "Fabrics" textures from the VisTeX database.

are usually not considered when parametric modeling of wavelet coefficients is of interest (see [3], [6], [7], [8] and [9]). In the following, we propose the use of Edgeworth expansions of order 4 for modeling the SWT approximation subbands.

When considering an Edgeworth expansion with order 4, we can capture the variance, skewness and kurtuosis similarity information between SWT approximations. The Edgeworth expansion of order p for a random variable X, absolutely continuous, with mean μ and standard deviation σ is given by:

$$f_p(x) = \frac{1}{\sqrt{2\pi}\sigma}\left(\sum_{r=0}^{p} \eta_r H_r(x)\right) e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad (2)$$

where H_r is the Chebyshev-Hermite polynomial of order r [10] and the coefficient η_r is a function of the r firsts cumulants of X. When $p = \infty$, the right hand side expression of the above equation is exactly the probability density function of X, under regularity assumptions on this function [11].

For modeling the SWT detail coefficients, we need to highlight the following facts: in the detail wavelet domain, stochastic processes tend to yield distributions that are regular with respect to the Gaussian distribution (see the literature on the wavelet transforms of stochastic processes, among which references [1] and [2] concern central limit theorems for wavelet decompositions). In contrast, geometrically regular functions tend to yield sparse distributions in the detail wavelet domain [12], and significant coefficients are those with large amplitudes (extremes). These properties suggest using:

- distributions that are regular with respect to the Gaussian distributions, such as the Generalized Gaussian densities, for approximating the distributions of stochastic textures.

- heavy tailed distributions that make possible a good fit to extremes of the sparse coefficients issued from the wavelet decomposition of regular textures.

Furthermore, when dealing with the whole database, we need the availability of a family of distribution functions that will realize a

kind of compromise in terms of heavy tails and regularity with respect to the Gaussian distributions. The Weibull distributions are such a family.

With respect to the above considerations, modeling the detail wavelet coefficients will be addressed by using Generalized Gaussian, Pareto and Weibull distributions (model validation is omitted in this short paper and the reader is asked to refer to [3] concerning how to perform model validation in CBIR experiments).

The Generalized Gaussian, GG, distribution with *scale* $\alpha > 0$ and *shape* $\beta > 0$ is defined by:

$$f_{\alpha,\beta}(x) = \frac{\beta}{2\alpha\Gamma(1/\beta)} e^{-(\frac{|x|}{\alpha})^\beta}, \tag{3}$$

for every real value x, where Γ is the standard Gamma function given for $z > 0$ by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t$.

The Pareto $(\lambda, 1)$-family, PRT, with distributions indexed by a *shape* parameter $\lambda > 0$ is given for $x \geqslant 0$ by:

$$f_\lambda(x) = \lambda (1 + x)^{-\lambda-1}. \tag{4}$$

The Weibull, WBL, distribution with *scale* $a > 0$ and *shape* $b > 0$ is defined for $x > 0$ by:

$$f_{a,b}(x) = \frac{b}{a} \left(\frac{x}{a}\right)^{b-1} e^{-(\frac{x}{a})^b}. \tag{5}$$

3.2. Similarity measurements between parametric models

We use as similarity measure between two random variables X_1 and X_2 having probability distribution functions f_{X_1} and f_{X_2}, the symmetric Kullback-Leibler divergence defined by:

$$\mathcal{K}(X_1, X_2) = \mathcal{K}(X_1||X_2) + \mathcal{K}(X_2||X_1), \tag{6}$$

$$\text{with} \quad \mathcal{K}(X_i||X_j) = \int_{\mathbb{R}} f_{X_i}(x) \log \frac{f_{X_i}(x)}{f_{X_j}(x)} dx, \quad i, j = 1, 2.$$

Let $\kappa_p[X_i]$ denotes the p-th cumulant of the random variable X_i. From [13], the Kullback-Leibler divergence between X_1 and X_2 can be approximated by:

$$\mathcal{K}(X_1||X_2) \approx \frac{1}{12} \frac{\kappa_3^2[X_1]}{\kappa_2^2[X_1]} + \frac{1}{2}\left(\beta^2 - 2\log\beta - 1 + \alpha^2\right)$$
$$+ (b_1 + b_2 + b_3)$$
$$+ \frac{1}{36} \frac{\kappa_3[X_1]\kappa_3[X_2]}{\kappa_2^2[X_2]}\left(\frac{1}{\kappa_2[X_2]} - \alpha^2 + 9\kappa_2[X_2]\right)$$

with:

$$\alpha = \frac{1}{\sqrt{\kappa_2[X_2]}}(\kappa_1[X_1] - \kappa_1[X_2]), \qquad \beta = \sqrt{\frac{\kappa_2[X_1]}{\kappa_2[X_2]}},$$

and

$$b_1 = \frac{\kappa_3^2[X_2]}{6\kappa_2^3[X_2]}\left(\beta^3\left(\alpha^3 + 3\alpha\right) - 3\beta\right),$$

$$b_2 = \frac{\kappa_4[X_2]}{24\kappa_2^2[X_2]}\left(\beta^4\left(\alpha^4 + 6\alpha^2 + 3\right) - 6\beta^2\left(\alpha^2 + 1\right) + 3\right),$$

$$b_3 = \frac{\kappa_3^2[X_2]\kappa_2^3[X_2]}{72\kappa_2^3[X_2]}\left(\beta^6\left(\alpha^6 + 15\alpha^4 + 45\alpha^2 + 15\right)\right.$$
$$- 15\beta^4\left(\alpha^4 + 6\alpha^2 + 3\right)$$
$$\left. + 45\beta^2\left(\alpha^2 + 1\right) - 15\right).$$

For the GG distributions, the Kullback-Leibler divergence is given in [7]:

$$\mathcal{K}(X_1||X_2) = \log\left(\frac{\beta_1}{\beta_2}\frac{\alpha_2}{\alpha_1}\frac{\Gamma(1/\beta_2)}{\Gamma(1/\beta_1)}\right) + \left(\frac{\alpha_1}{\alpha_2}\right)^{\beta_2}\frac{\Gamma\left(\frac{1+\beta_2}{\beta_1}\right)}{\Gamma(1/\beta_1)} - \frac{1}{\beta_1}. \tag{7}$$

Since we consider the symmetric version of the Kullback-Leibler divergence, we have from Eqs. (6) and (7) that:

$$\mathcal{K}(X_1, X_2) = \left(\frac{\alpha_1}{\alpha_2}\right)^{\beta_2}\frac{\Gamma\left(\frac{1+\beta_2}{\beta_1}\right)}{\Gamma(1/\beta_1)} + \left(\frac{\alpha_2}{\alpha_1}\right)^{\beta_1}\frac{\Gamma\left(\frac{1+\beta_1}{\beta_2}\right)}{\Gamma(1/\beta_2)} - \frac{\beta_1 + \beta_2}{\beta_1\beta_2}. \tag{8}$$

For PRT distributions, the Kullback-Leibler divergence is given in [14]. We have that the symmetric version of this divergence is

$$\mathcal{K}(X_1, X_2) = \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2} + \log\frac{\lambda_2}{\lambda_1} + \log\frac{\lambda_1}{\lambda_2} - 2. \tag{9}$$

For WBL distributions, the Kullback-Leibler divergence is given in [6]. We have that the symmetric version of this divergence is

$$\mathcal{K}(X_1, X_2) = \Gamma\left(1 + \frac{k_2}{k_1}\right)\left(\frac{\lambda_1}{\lambda_2}\right)^{k_2} + \Gamma\left(1 + \frac{k_1}{k_2}\right)\left(\frac{\lambda_2}{\lambda_1}\right)^{k_1}$$
$$+ (k_1 - k_2)\log\frac{\lambda_1}{\lambda_2} + \mathfrak{e}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1} - 2\right) - 2.$$

where $\mathfrak{e}$ is the Euler-Mascheroni constant.

As mentioned above, wavelets tend to distribute many stochastic processes as *iid* random sequences when the wavelet decomposition level and filter order are large enough. These statistical properties make adequate the choice of the following cumulative similarity measure upon the distribution models associated with the SWT coefficients:

$$\mathcal{K}(I, \Upsilon) = \mathcal{K}(f_{c_{J,0}}[I], f_{c_{J,0}}[\Upsilon]) + \sum_{\substack{j \in \{1,2...,J\} \\ n \in \{1,2,3\}}} \mathcal{K}(f_{c_{j,n}}[I], f_{c_{j,n}}[\Upsilon]),$$

where $\mathcal{K}$ denotes the symmetric Kullback-Leibler divergence, I and Υ are two arbitrary textures and $f_{c_{j,n}}[I]$, $f_{c_{j,n}}[\Upsilon]$ are the *pdfs* used for modeling the subband (j, n) SWT coefficients of I and Υ respectively.

4. EXPERIMENTAL RESULTS

Experimental tests concern 40 texture classes of the VisTeX database. The database structuring for these classes is given, in terms of stochastic versus regular textures, in Table 1 (see Section 2 for details on the structuring method). The structuring yields a stochastic metaclass composed with 22 texture classes and a regular metaclass composed with 18 texture classes.

Any given texture class is composed with 16 images obtained by splitting every large texture image in 16 non-overlapping subimages. Summarizing, we have a test database $\mathcal{T}$ of 640 images, among which, 352 images forming a database structure $\mathcal{T}_1$ are issued from a stochastic class; whereas the 288 remaining textures constitute a database structure $\mathcal{T}_2$ associated with regular texture classes, with $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$.

We then run CBIR from parametric modeling and similarity measurements, as described in Section 3. Experimental tests are performed independently on the tree database structures $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}$. For a given structure, performance measurements concern the retrieval rates, when a query is any subimage of the structure under

Table 1. Texture-specific retrieval results for 40 textures in the Vis-TeX database. Experimental tests are performed separately on the stochasticity structures. Stochastic texture classes are given in red whereas regular classes are colored in blue.

Texture	GG	WBL	PRT	Texture	GG	WBL	PRT
Bark.00	69.92	68.36	78.13	Food.08	99.61	100	86.33
Bark.06	85.55	85.94	64.06	Gras.01	98.83	98.83	53.13
Bark.08	69.53	68.36	56.64	Leav.08	82.03	83.20	80.86
Bark.09	48.05	47.27	73.05	Leav.10	64.84	63.67	77.34
Bric.01	98.83	98.83	76.56	Leav.11	73.05	72.66	87.50
Bric.04	84.77	83.20	88.28	Leav.12	98.05	97.27	53.91
Bric.05	92.97	89.84	83.98	Leav.16	72.27	71.48	87.11
Buil.09	76.56	97.66	88.28	Meta.00	83.20	82.42	59.38
Fabr.00	94.92	91.80	78.13	Meta.02	100	100	86.33
Fabr.04	89.84	87.89	84.38	Misc.02	96.09	95.70	56.64
Fabr.07	98.05	98.05	75.39	Sand.00	96.48	97.66	51.56
Fabr.09	100	100	80.47	Ston.01	73.83	74.61	78.13
Fabr.11	92.58	92.58	57.81	Ston.04	93.75	92.97	49.22
Fabr.14	100	100	89.84	Terr.10	63.28	62.50	88.28
Fabr.15	92.97	92.97	57.03	Tile.01	62.11	61.72	91.41
Fabr.17	92.58	96.09	85.16	Tile.04	99.61	99.61	94.14
Fabr.18	94.92	91.80	47.27	Tile.07	99.22	98.83	83.98
Flow.05	68.36	66.41	80.86	Wate.05	100	100	56.25
Food.00	100	100	87.50	Wood.01	61.33	61.33	75.78
Food.05	81.64	81.64	72.27	Wood.02	100	100	67.58

consideration. Retrieval rates per class are given in Table 1 concerning $\mathcal{T}_1$ and $\mathcal{T}_2$. Average retrieval rates per structures structures $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}$ are given in Table 2 for comparison purpose.

From these experimental results, we can conclude that the retrieval is more concise when the search focuses either on $\mathcal{T}_1$ or on $\mathcal{T}_2$ than on the whole structure $\mathcal{T}$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ have low cardinality, the structuring also eases the search. In addition, from Table 2 and when comparing the role played by the distribution type on the metaclass, it follows that the more relevant family is:

- the GG family for modeling the stochastic textures,
- the PRT family for modeling the regular textures,
- the WBL family for modeling the whole database containing both regular and stochastic textures.

The above remarks confirm the suitability of separating a heterogeneous database into structures with approximately the same statistical properties.

Table 2. Average values of texture-specific retrieval for the whole database $\mathcal{T}$, the database composed of *stochastic textures* $\mathcal{T}_1$ and the database composed of *regular textures* $\mathcal{T}_2$, with $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}$. Experimental results performed without stochastic structuration (blind approach) are given for comparison purpose.

Stochastic textures ($\mathcal{T}_1$)					
Blind approach			Stochastic structuring		
GG	WBL	PRT	GG	WBL	PRT
88.12	87.82	66.05	90.45	90.02	66.67
Regular textures ($\mathcal{T}_2$)					
Blind approach			Stochastic structuring		
GG	WBL	PRT	GG	WBL	PRT
78.95	79.60	83.18	81.10	81.81	83.51
Whole texture database ($\mathcal{T}$)					
Blind approach			Stochastic structuring		
GG	WBL	PRT	GG	WBL	PRT
83.99	84.12	73.76	86.24	86.33	74.25

5. CONCLUSION

In this paper, we have first investigated structuring large databases with respect to stochasticity consideration. The structuring is derived from the Kolmogorov stochasticity parameter applied in the stationary wavelet domain. This structuring yields stochastic versus regular texture metaclasses. Then, we have addressed content-based image retrieval from Edgeworth, Generalized Gaussian, Pareto and Weibull distributions for modeling the coefficients of the stationary wavelet transform. Experimental tests highlight the relevancy of the Pareto distribution for modeling the regular metaclass: Regular textures are sparsely distributed in the wavelet domain and the behavior of their large (significant) coefficients can be captured by using the tail of the Pareto distribution. Concerning the stochastic textures, those textures that have regular distributions with respect to the Gaussian distribution in the wavelet domain, they tend to be well described by the Generalized Gaussian family. The relevance of data structuring with respect to stochasticity has been emphasized by content-based image retrieval experiments: 1) the computational load on the metaclasses is reduced by half to a quarter with respect to that involved for the whole database and 2) performance are increased by restricting the search to the appropriate metaclass. Prospects concern refinements of the structuring by setting different stochasticity bounds on the Kolmogorov parameter.

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