Central Limit Theorems for Wavelet Packet Decompositions of Stationary Random Processes

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***Abstract*—This paper provides central limit theorems for the wavelet packet decomposition of stationary band-limited random processes. The asymptotic analysis is performed for the sequences of the wavelet packet coefficients returned at the nodes of any given path of the M-band wavelet packet decomposition tree. It is shown that if the input process is strictly stationary, these sequences converge in distribution to white Gaussian processes when the resolution level increases, provided that the decomposition filters satisfy a suitable property of regularity. For any given path, the variance of the limit white Gaussian process directly relates to the value of the input process power spectral density at a specific frequency.**

***Index Terms*—Wavelet transforms, Band-limited stochastic processes, Spectral analysis.**

I. Introduction

This paper addresses the statistical properties of the M-Band Discrete Wavelet Packet Transform, hereafter abbreviated as M-DWPT. Specifically, an asymptotic analysis is given for the correlation structure and the distribution of the M-Band wavelet packet coefficients of stationary random processes.

In [1] and [2], such a study is carried out without analysing the role played by the path followed in the M-DWPT tree and that of the wavelet decomposition filters. In contrast, this correspondence paper emphasizes that, given a path of the M-DWPT, the sequence of the M-Band wavelet packet coefficients obtained at resolution j in this path converges, in a distributional sense specified below, to a discrete white Gaussian process, when j tends to infinity. The variance of the limit process depends on 1) the path followed in the M-DWPT tree, 2) the wavelet decomposition filters and 3) the value, at a specific frequency, of the power spectral density (spectrum) of the input random process. This analysis is presented for the Shannon M-DWPT filters and standard families of paraunitary filters that converge to the Shannon paraunitary filters.

II. Preliminary results

A. Tree decomposition and path representations

Throughout, M is a natural number larger than or equal to 2, j and n always refer to non-negative integers. As usual, $\mathbb{N} = \{1, 2, \dots\}$ stands for the set of natural numbers and $\mathbb{Z}$ for the set of integers. The tree $\mathcal{T}$ considered below is constructed as follows: $\mathcal{T}$ has a root (or starting) node $\mathbf{U} \equiv \mathbf{W}_{0,0}$ and double indexed children nodes $\mathbf{W}_{j,n}$, where $j \geqslant 1$ and $n \in \{0, 1, \dots, M^j - 1\}$ for every fixed j. In this tree decomposition, the children of $\mathbf{W}_{j,n}$ are defined to be $\mathbf{W}_{j+1,Mn+m}$, where $m = 0, 1, \dots, M-1$. The index j will be referred as the decomposition level and the index n as the frequency index. We use the notation $\mathcal{T} = \left(\mathbf{U}, \{\mathbf{W}_{j,n}\}_{j\geqslant 1, n\in\{0,1,\dots,M^j-1\}}\right)$. In what follows, a path is any sequence of spaces $\left(\mathbf{U}, \{\mathbf{W}_{j,n_j}\}_{j\geqslant 1}\right)$ such that $\mathbf{W}_{j,n_j}$ is a child of $\mathbf{W}_{j-1,n_{j-1}}$ for every $j \geqslant 1$, with $n_0 = 0$ by convention.

Let $\mathcal{P}$ be a given path of $\mathcal{T}$. This path is described by a sequence of nodes (spaces) where the frequency index is

$$n_j = M n_{j-1} + m_j, \tag{1}$$

for $j \geqslant 1$, with $m_j \in \{0, 1, \dots, M-1\}$. Therefore, in path $\mathcal{P}$ and at each decomposition level j, the frequency index is $n_j = \sum_{\ell=1}^{j} m_\ell M^{j-\ell} \in \{0, 1, \dots, M^j - 1\}$. By construction, path $\mathcal{P}$ can be associated with a unique M-ary sequence $(m_\ell)_{\ell\in\mathbb{N}}$ of elements of $\{0, 1, \dots, M-1\}$. On the other hand, any frequency index $n \in \{0, 1, \dots, M^j - 1\}$ at decomposition level $j \geqslant 1$ can be associated with a unique finite subsequence $(m_1, m_2, \dots, m_j)$ of elements of $\{0, 1, \dots, M-1\}$ such that $n = \sum_{\ell=1}^{j} m_\ell M^{j-\ell}$. This unique subsequence will hereafter be called the M-ary subsequence associated with the pair (j, n). With the terminology and notation introduced above, the sole sequence of nodes $(j, n_j)_{j\geqslant 1}$ that specifies path $\mathcal{P}$ is such that the M-ary subsequence associated with (j, n_j) results from the concatenation of the M-ary subsequence associated with $(j-1, n_{j-1})$ with the unique value m_j such that Eq. (1) holds true.

In what follows, $\mathcal{T}$ is an M-DWPT tree whose nodes are the orthogonal nested functional subspaces generated from a root space $\mathbf{U} \subset L^2(\mathbb{R})$ by using wavelet paraunitary filters with impulse responses $h_m, m = 0, 1, 2, \dots, M-1$. For further details about the computation and the properties of M-DWPT filters, the reader is asked to refer to [3]. The Fourier transform of the paraunitary filter with impulse response $h_m, m = 0, 1, 2, \dots, M-1$, is hereafter defined by

$$H_m(\omega) = \frac{1}{\sqrt{M}} \sum_{\ell\in\mathbb{Z}} h_m[\ell] \exp\left(-i\ell\omega\right). \tag{2}$$

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B. General formulas on the M-DWPT

Let Φ be a function such that $\{\tau_k\Phi \ : k \in \mathbb{Z}\}$ is an orthonormal system of $L^2(\mathbb{R})$, where $\tau_k\Phi \ : t \longmapsto \Phi(t-k)$. Let $\mathbf{U}$ be the closure of the space spanned by this orthonormal system. With notation similar to [4], [5], the M-DWPT decomposition of the function space $\mathbf{U}$ involves splitting $\mathbf{U}$ into M orthogonal subspaces (an easy extension [6, Lemma 10.5.1] established for the standard DWPT) so that

$$\mathbf{U} = \bigoplus_{m=0}^{M-1} \mathbf{W}_{1,m}$$

and recursively applying the following splitting

$$\mathbf{W}_{j,n} = \bigoplus_{m=0}^{M-1} \mathbf{W}_{j+1,Mn+m},$$

for every natural number j and every $n = 0,1,2,\ldots,M^j-1$.

Given $j \geqslant 0$ and $n \in \{0,1,\ldots,M^j-1\}$, let us consider the *wavelet packet space* $\mathbf{W}_{j,n}$ located at node (j,n) of the wavelet packet tree. This function space is the closure of the space spanned by the orthonormal set of the *wavelet packet functions* $\{\psi_{j,n,k} : k \in \mathbb{Z}\}$, with

$$\psi_{j,n,k}(t) = \psi_{j,n}(t - M^j k), \tag{3}$$

where the sequence $(\psi_{p,q})_{p,q}$ is recursively defined by putting $\psi_{0,0} = \Phi$ and setting, for any $p \geqslant 0$, any $q \geqslant 0$ and any $m \in \{0,1,\ldots,M-1\}$,

$$\psi_{p+1,Mq+m}(t) = \sum_{\ell\in\mathbb{Z}} h_m[\ell]\psi_{p,q}(t - M^p\ell). \tag{4}$$

Each $\psi_{j,n}$, where $j \geqslant 1$ and $n \in \{0,1,\ldots,M^j-1\}$, is thus obtained from Φ and the particular sequence of filters $(h_{m_1}, h_{m_2}, \ldots, h_{m_j})$ where $(m_1, m_2, \ldots, m_j)$ is the M-ary subsequence associated with (j,n).

The standard formulas [4, p. 324, Eqs. (8.10) and (8.11)] are obtained by setting $M = 2$ above. It is worth emphasizing that Φ is not necessarily the scaling function associated with the low-pass filter h_0. In other words, the M-DWPT applies to the general case where the input decomposition space $\mathbf{U}$ is not necessarily the space generated by the translated versions of the scaling function associated with h_0. We thus can fix an input space and decompose it by using different types of wavelet paraunitary filters. This is exactly what is done in sections III and IV where the input functional space is always the standard Paley-Wiener space but different M-DWPT filters are used to decompose it. In the particular case where Φ is the scaling function associated with h_0, the standard scaling equation [4, p. 228, Eq. (7.28)] implies that $\psi_{1,0}(t) = (1/\sqrt{2})\Phi(t/2)$.

Given $f \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$, let $\mathcal{F}f$ henceforth stands for the Fourier transform of f, where $\mathcal{F}f$ is given by

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}} f(t)\exp(-i\omega t)\,\mathrm{d}t$$

if $f \in L^1(\mathbb{R})$. Given any $j \geqslant 1$ and any $n \in \{0,1,\ldots,M^j-1\}$, a straightforward recurrence based on the Fourier transform of Eq. (4) leads to

$$\mathcal{F}\psi_{j,n}(\omega) = M^{j/2}\left[\prod_{\ell=1}^{j} H_{m_\ell}(M^{\ell-1}\omega)\right]\mathcal{F}\Phi(\omega), \tag{5}$$

where $(m_1, m_2, \ldots, m_j)$ is the M-ary subsequence associated with (j,n). This standard result will prove useful in the sequel.

C. Shannon M-DWPT and the Paley-Wiener space of π band-limited functions

The Shannon M-DWPT filters are hereafter denoted h_m^{S} for $m = 0,1,\ldots,M-1$. These filters are ideal low-pass, band-pass and high-pass filters. We have

$$H_m^{\mathsf{S}}(\omega) = \sum_{\ell\in\mathbb{Z}} \mathbb{1}_{\Delta_m}(\omega - 2\pi\ell), \tag{6}$$

where $\mathbb{1}_K$ denotes the indicator function of a given set K: $\mathbb{1}_K(x) = 1$ if $x \in K$ and $\mathbb{1}_K(x) = 0$, otherwise, and

$$\Delta_m = \left[-\frac{(m+1)\pi}{M}, -\frac{m\pi}{M}\right] \cup \left[\frac{m\pi}{M}, \frac{(m+1)\pi}{M}\right].$$

The scaling function Φ^{S} associated with these filters is defined for every $t \in \mathbb{R}$ by $\Phi^{\mathsf{S}}(t) = \mathsf{sinc}(t) = \sin(\pi t)/\pi t$ with $\Phi^{\mathsf{S}}(0) = 1$. The Fourier transform of this scaling function is

$$\mathcal{F}\Phi^{\mathsf{S}} = \mathbb{1}_{[-\pi,\pi]}. \tag{7}$$

The closure $\mathbf{U}^{\mathsf{S}}$ of the space spanned by the orthonormal system $\{\tau_k\Phi^{\mathsf{S}} : k \in \mathbb{Z}\}$ is then the Paley-Wiener (PW) space of those elements of $L^2(\mathbb{R})$ that are π band-limited in the sense that their Fourier transform is supported within $[-\pi,\pi]$.

Let X be any band-limited Wide-Sense Stationary (WSS) random process whose spectrum is supported within $[-\pi,\pi]$. We have (see [7, Appendix D])

$$X[k] = \int_{\mathbb{R}} X(t)\Phi^{\mathsf{S}}(t-k)\,\mathrm{d}t, \tag{8}$$

so that $\mathbf{U}^{\mathsf{S}}$ is the natural representation space of such a process. Any M-DWPT of X can thus be initialized with the samples $X[k]$, $k \in \mathbb{Z}$.

Now, let us consider the Shannon M-DWPT of the PW space $\mathbf{U}^{\mathsf{S}}$. The wavelet packet functions $\psi_{j,n}^{\mathsf{S}}$ of this M-DWPT can be computed by means of Eq. (4) with $\Phi = \Phi^{\mathsf{S}}$ and $h_m = h_m^{\mathsf{S}}$, $m = 0,1,\ldots,M-1$. The Fourier transforms of these wavelet packet functions are given by proposition 1 below, which extends [4, Proposition 8.2, p. 328] since the latter follows from the former with $M = 2$.

Proposition 1: For $j \geqslant 0$ and $n \in \{0,1,\ldots,M^j-1\}$, we have

$$\mathcal{F}\psi_{j,n}^{\mathsf{S}} = \mathbb{1}_{\Delta_{j,G(n)}} \tag{9}$$

where, for any non-negative integer k,

$$\Delta_{j,k} = \left[-\frac{(k+1)\pi}{M^j}, -\frac{k\pi}{M^j}\right] \cup \left[\frac{k\pi}{M^j}, \frac{(k+1)\pi}{M^j}\right] \tag{10}$$

and G is the map defined by $G(0) = 0$ and recursively setting, for $m = 0,1,\ldots,M-1$ and $\ell = 0,1,2,\ldots$,

$$G(M\ell+m) = \begin{cases} MG(\ell) + m & \text{if } G(\ell) \text{ is even,} \\ MG(\ell) - m + M - 1 & \text{if } G(\ell) \text{ is odd.} \end{cases} \tag{11}$$

Proof: A routine exercice based on Eqs. (6), (7) and the recursive definition of the wavelet packet functions. ■

In the rest of the paper, we set, for any pair (j,k) of non-negative integers,

$$\Delta^{+}_{j,k} = \left[\frac{k\pi}{M^j}, \frac{(k+1)\pi}{M^j}\right]. \qquad (12)$$

III. Asymptotic analysis for the autocorrelation functions of the M-DWPT of second-order WSS random processes

Let X denote a zero-mean second-order real random process assumed to be continuous in quadratic mean. The autocorrelation function of X, denoted by R, is defined by $R(t,s) = \mathbb{E}[X(t)X(s)]$. Given $j \geqslant 1$ and $n \in \{0,1,\ldots,M^j-1\}$, the projection of X on the wavelet packet space $\mathbf{W}_{j,n}$ yields a sequence of random variables, the wavelet packet *coefficients* of X:

$$c_{j,n}[k] = \int_{\mathbb{R}} X(t)\psi_{j,n,k}(t)\,\mathrm{d}t, \quad k \in \mathbb{Z}, \qquad (13)$$

provided that $\iint_{\mathbb{R}^2} R(t,s)\psi_{j,n,k}(t)\psi_{j,n,k}(s)\,\mathrm{d}t\,\mathrm{d}s < \infty$, which will be assumed in the rest of the paper since commonly used wavelet functions are compactly supported or have sufficiently fast decay. The sequence given by Eq. (13) defines the discrete random process $c_{j,n} = (c_{j,n}[k])_{k\in\mathbb{Z}}$ of the wavelet packet *coefficients* of X at resolution level j and for frequency index n.

A. Problem formulation

Let $R_{j,n}$ stand for the autocorrelation function of the random process $c_{j,n}$. We have

$$\begin{aligned} R_{j,n}[k,\ell] &= \mathbb{E}\big[c_{j,n}[k]c_{j,n}[\ell]\big] \\ &= \iint_{\mathbb{R}^2} R(t,s)\psi_{j,n,k}(t)\psi_{j,n,\ell}(s)\,\mathrm{d}t\,\mathrm{d}s. \end{aligned} \qquad (14)$$

If X is WSS, we write $R(t,s) = R(t-s)$ with some usual and slight abuse of language. From Eq. (14), it follows that

$$R_{j,n}[k,\ell] = \iint_{\mathbb{R}^2} R(t)\psi_{j,n,k}(t+s)\psi_{j,n,\ell}(s)\,\mathrm{d}t\,\mathrm{d}s. \qquad (15)$$

In the sequel, the spectrum γ of X, that is, the Fourier transform of R, is assumed to exist. By taking into account that the Fourier transform of $\psi_{j,n,k}$ is

$$\mathcal{F}\psi_{j,n,k}(\omega) = \exp\left(-iM^j k\omega\right)\mathcal{F}\psi_{j,n}(\omega),$$

by using Fubini's theorem and Parseval's equality, we derive from Eq. (15) that $c_{j,n}$ is WSS. For any $k,\ell \in \mathbb{Z}$, and with the same abuse of language as above, the value $R_{j,n}[k,\ell]$ of the autocorrelation function of the discrete random process $c_{j,n}$ is $R_{j,n}[k-\ell]$ with

$$R_{j,n}[k] = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma(\omega)|\mathcal{F}\psi_{j,n}(\omega)|^2 \exp\left(iM^j k\omega\right)\mathrm{d}\omega. \qquad (16)$$

The purpose of the next section is then to analyse the behaviour of $R_{j,n}$ when j tends to ∞, in the case of the Shannon filters and some families of filters that converge to the Shannon filters. From now on, the input decomposition space is assumed to be the PW space $\mathbf{U}^{\mathsf{S}}$.

B. Asymptotic decorrelation

Consider the Shannon M-DWPT, that is, the decomposition of $\mathbf{U}^{\mathsf{S}}$ associated with the Shannon M-DWPT filters $(h^{\mathsf{S}}_m)_{m=0,1,\ldots,M-1}$. With the same notation and terminology as in section II, let $(m_\ell)_{\ell\in\mathbb{N}}$ be an M-ary sequence of elements of $\{0,1,\ldots,M-1\}$ and $\mathcal{P} = (\mathbf{U}^{\mathsf{S}}, \{\mathbf{W}^{\mathsf{S}}_{j,n_j}\}_{j\in\mathbb{N}})$ be the path associated with this subsequence in the Shannon M-DWPT decomposition tree. It follows from proposition 1 that the support of $\mathcal{F}\psi^{\mathsf{S}}_{j,n_j}$ is $\Delta_{j,G(n_j)}$. For $j \in \mathbb{N}$, the sets $\Delta^{+}_{j,G(n_j)}$ are nested closed intervals whose diameters tend to 0. Therefore, their intersection contains only one point henceforth denoted by $\omega_{\mathcal{P}}$. It then follows from Eq. (12) that

$$\omega_{\mathcal{P}} = \lim_{j\to+\infty} \frac{G(n_j)\pi}{M^j}. \qquad (17)$$

Let X be some zero-mean second-order WSS random process, continuous in quadratic mean, with spectrum γ. The autocorrelation function R^{S}_{j,n_j} resulting from the projection of X on $\mathbf{W}^{\mathsf{S}}_{j,n_j}$ derives from Eq. (16) and is given by

$$R^{\mathsf{S}}_{j,n_j}[k] = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma(\omega)|\mathcal{F}\psi^{\mathsf{S}}_{j,n_j}(\omega)|^2 \exp\left(iM^j k\omega\right)\mathrm{d}\omega. \qquad (18)$$

From Eqs. (9) and (18) and by taking into account that γ is even, as the Fourier transform of the even function R, it follows that

$$R^{\mathsf{S}}_{j,n_j}[k] = \frac{M^j}{\pi}\int_{\Delta^{+}_{j,G(n_j)}} \gamma(\omega)\cos\left(M^j k\omega\right)\mathrm{d}\omega. \qquad (19)$$

where $\Delta^{+}_{j,G(n_j)}$ is given by Eq. (12). When X satisfies some additional assumptions, the following theorem 1 states that the Shannon M-DWPT of X yields coefficients that tend to be decorrelated when j tends to infinity. One of these additional assumptions is that X is band-limited in the sense that its spectrum is supported within $[-\pi,\pi]$.

Theorem 1: Let X be a zero-mean second-order WSS random process, continuous in quadratic mean. Assume that the spectrum γ of X is an element of $L^\infty(\mathbb{R})$ and is supported within $[-\pi,\pi]$. Let $\mathcal{P} = (\mathbf{U}^{\mathsf{S}}, \{\mathbf{W}^{\mathsf{S}}_{j,n_j}\}_{j\in\mathbb{N}})$ be a Shannon M-DWPT decomposition path.

If the spectrum γ of X is continuous at point $\omega_{\mathcal{P}}$, then

$$\lim_{j\to+\infty} R^{\mathsf{S}}_{j,n_j}[k] = \gamma(\omega_{\mathcal{P}})\delta[k] \qquad (20)$$

uniformly in $k \in \mathbb{Z}$, where R^{S}_{j,n_j} is the autocorrelation function of the coefficients resulting from the projection of X on $\mathbf{W}^{\mathsf{S}}_{j,n_j}$.

Proof: The proof is an easy generalisation of that of [7, Proposition 1], which concerns the standard wavelet packet transform ($M = 2$). ■

The foregoing theorem is mainly of theoretical interest since the Shannon M-DWPT filters have infinite supports and are not really suitable for practical purpose. In order to obtain a result of the same type for filters of practical interest, the M-DWPT of $\mathbf{U}^{\mathsf{S}}$ is now assumed to be performed by using decomposition filters of order r, $h^{[r]}_m$, $m = 0,1,\ldots,M-1$, whose Fourier transforms $H_m^{[r]}$ are defined by Eq. (2) and such that

$$\lim_{r\to\infty} H_m^{[r]} = H_m^{\mathsf{S}} \quad \text{(a.e.)}. \tag{21}$$

Similarly to above, $\mathbf{W}_{j,n}^{[r]} \subset \mathbf{U}^{\mathsf{S}}$ henceforth stands for the wavelet packet space at node (j,n). The wavelet packet functions $\psi_{j,n}^{[r]}$ of the M-DWPT under consideration are now calculated by applying Eq. (4) with $\Phi = \Phi^{\mathsf{S}}$ and $h_m = h_m^{[r]}$, $m = 0, 1, \ldots, M-1$.

According to [8], the Daubechies filters satisfy Eq. (21) for $M = 2$ when r is the number of vanishing moments of the Daubechies wavelet function. According to [9], the Battle-Lemarié filters also satisfy Eq. (21) for $M = 2$ when r is the spline order of the Battle-Lemarié scaling function. The existence of such families for $M > 2$ remains an open issue to address in forthcoming work. However, it can reasonably be expected that general M-DWPT filters of the Daubechies or Battle-Lemarié type converge to the Shannon filters in the sense given above.

Theorem 2: Let X be a zero-mean second-order WSS random process, continuous in quadratic mean. Assume that the spectrum γ of X is an element of $L^\infty(\mathbb{R})$ and is supported within $[-\pi,\pi]$. Assume that the M-DWPT of the PW space $\mathbf{U}^{\mathsf{S}}$ is achieved by using decomposition filters $h_m^{[r]}$, $m = 0, 1, \ldots, M-1$, satisfying Eq. (21). Let $R_{j,n}^{[r]}$ stand for the autocorrelation function of the wavelet packet coefficients of X with respect to the wavelet packet space $\mathbf{W}_{j,n}^{[r]}$. We have

$$\lim_{r\to+\infty} R_{j,n}^{[r]}[k] = R_{j,n}^{\mathsf{S}}[k], \tag{22}$$

uniformly in $k \in \mathbb{Z}$, where $R_{j,n}^{\mathsf{S}}$ is given by Eq. (19).

Remark 1: Albeit straightforward, the following equalities are useful in the sequel. Let $(m_1, m_2, \ldots, m_j)$ be the M-ary subsequence associated with a given pair (j,n). From Eq. (5), the functions $\psi_{j,n}^{\mathsf{S}}$ that satisfy Eq. (4) for the Shannon M-DWPT of $\mathbf{U}^{\mathsf{S}}$ are such that

$$\mathcal{F}\psi_{j,n}^{\mathsf{S}}(\omega) = M^{j/2}\left[\prod_{\ell=1}^{j} H_{m_\ell}^{\mathsf{S}}(M^{\ell-1}\omega)\right]\mathcal{F}\Phi^{\mathsf{S}}(\omega). \tag{23}$$

In the same way, the functions $\psi_{j,n}^{[r]}$, involved in Eq. (4) for the decomposition of $\mathbf{U}^{\mathsf{S}}$ *via* the filters $h_m^{[r]}$ introduced above, satisfy the following equality

$$\mathcal{F}\psi_{j,n}^{[r]}(\omega) = M^{j/2}\left[\prod_{\ell=1}^{j} H_{m_\ell}^{[r]}(M^{\ell-1}\omega)\right]\mathcal{F}\Phi^{S}(\omega). \tag{24}$$

In addition, from Eqs. (21), (23) and (24) we have that

$$\lim_{r\to+\infty} \mathcal{F}\psi_{j,n}^{[r]} = \mathcal{F}\psi_{j,n}^{\mathsf{S}} \quad \text{(a.e.)}. \tag{25}$$

Proof: (of theorem 2). The autocorrelation function $R_{j,n}^{[r]}$ is given by Eq. (16) and is equal to

$$R_{j,n}^{[r]}[k] = \frac{1}{2\pi}\int_{\mathbb{R}} \gamma(\omega)|\mathcal{F}\psi_{j,n}^{[r]}(\omega)|^2 \exp\left(iM^j k\omega\right) \mathrm{d}\omega. \tag{26}$$

In addition, we have

$$\begin{aligned}&\left|R_{j,n}^{[r]}[k] - R_{j,n}^{\mathsf{S}}[k]\right| \\ &\quad\leqslant \frac{1}{2\pi}\int_{\mathbb{R}} |\gamma(\omega)| \left| |\mathcal{F}\psi_{j,n}^{[r]}(\omega)|^2 - |\mathcal{F}\psi_{j,n}^{\mathsf{S}}(\omega)|^2\right| \mathrm{d}\omega.\end{aligned} \tag{27}$$

From Eqs. (23) and (24) and by taking into account that $|H_{m_\ell}^{[r]}(\omega)|$ and $|H_{m_\ell}^{\mathsf{S}}(\omega)|$ are less than or equal to 1 (due to the paraunitarity of the M-DWPT filters), we obtain

$$\left| |\mathcal{F}\psi_{j,n}^{[r]}(\omega)|^2 - |\mathcal{F}\psi_{j,n}^{\mathsf{S}}(\omega)|^2\right| \leqslant 2M^j \left|\mathcal{F}\Phi^{\mathsf{S}}(\omega)\right|^2. \tag{28}$$

The result then derives from Eqs. (25), (27), (28) and Lebesgue's dominated convergence theorem. ■

IV. Central limit theorems

In this section, we consider a zero-mean real random process X that has finite cumulants and polyspectra. In what follows, q is a natural number. Let us denote by

$$\mathsf{cum}(t, s_1, s_2, \ldots, s_q) = \mathsf{cum}\{X(t), X(s_1), X(s_2), \ldots, X(s_q)\}$$

the cumulant of order $q+1$ of X. The above cumulant is hereafter assumed to belong to $L^2(\mathbb{R}^{q+1})$ and to be finite (see [10, Proposition 1] for a discussion about the existence of this cumulant). With the notation introduced so far, the cumulant of order $q+1$ of the random process $c_{j,n}$ has the integral form given by

$$\begin{aligned}&\mathsf{cum}_{j,n}[k, \ell_1, \ell_2, \ldots, \ell_q] \\ &\quad= \mathsf{cum}\left\{c_{j,n}[k]c_{j,n}[\ell_1]c_{j,n}[\ell_2]\ldots c_{j,n}[\ell_q]\right\} \\ &\quad= \int_{\mathbb{R}^{q+1}} \mathrm{d}t\,\mathrm{d}s_1\,\mathrm{d}s_2\ldots\mathrm{d}s_q\,\mathsf{cum}(t, s_1, s_2, \ldots, s_q)\psi_{j,n,k}(t) \\ &\qquad\qquad \psi_{j,n,\ell_1}(s_1)\psi_{j,n,\ell_2}(s_2)\ldots\psi_{j,n,\ell_q}(s_q).\end{aligned} \tag{29}$$

If X is assumed to be strictly stationary so that $\mathsf{cum}(t, t+t_1, t+t_2, \ldots, t+t_q) = \mathsf{cum}(t_1, t_2, \ldots, t_q)$, then $c_{j,n}$ is a strictly stationary random process with cumulants $\mathsf{cum}_{j,n}[k, k+k_1, k+k_2, \ldots, k+k_q] = \mathsf{cum}_{j,n}[k_1, k_2, \ldots, k_q]$. Assume also that X has a polyspectrum $\gamma_q(\omega_1, \omega_2, \ldots, \omega_q) \in L^\infty(\mathbb{R}^q)$ for every natural number q and every $(\omega_1, \omega_2, \ldots, \omega_q) \in \mathbb{R}^q$. The polyspectrum is the Fourier transform of the cumulant $\mathsf{cum}(t_1, t_2, \ldots, t_q)$. When $q = 1$, γ_1 is the spectrum of X and is simply denoted γ as in section III. Then, after some routine algebra, Eq. (29) reduces to:

$$\begin{aligned}&\mathsf{cum}_{j,n}[k_1, k_2, \ldots, k_q] \\ &\quad= \frac{1}{(2\pi)^q}\int_{\mathbb{R}^q} \mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\ldots\mathrm{d}\omega_q \\ &\qquad\qquad \exp\left(-iM^j(k_1\omega_1 + k_2\omega_2 + \ldots + k_q\omega_q)\right) \\ &\qquad\qquad \gamma_q(-\omega_1, -\omega_2, \ldots, -\omega_q) \\ &\qquad\qquad \mathcal{F}\psi_{j,n}(-\omega_1 - \omega_2 - \ldots - \omega_q) \\ &\qquad\qquad \mathcal{F}\psi_{j,n}(\omega_1)\mathcal{F}\psi_{j,n}(\omega_2)\ldots\mathcal{F}\psi_{j,n}(\omega_q).\end{aligned} \tag{30}$$

We then have:

Theorem 3: Let X be a zero-mean second-order strictly stationary random process, continuous in quadratic mean. Assume that the polyspectrum γ_q of X is an element of

$L^\infty(\mathbb{R}^q)$ for any $q \in \mathbb{N}$ and that the spectrum γ of X is supported within $[-\pi,\pi]$. Consider the Shannon M-DWPT of the PW space $\mathbf{U}^{\mathsf{S}}$. If $q \geqslant 2$, the cumulant of order $q+1$, $\mathsf{cum}^{\mathsf{S}}_{j,n}[k_1,k_2,\dots,k_q]$, of the discrete random process resulting from the projection of X on $\mathbf{W}^{\mathsf{S}}_{j,n}$ satisfies

$$\lim_{j\to+\infty} \mathsf{cum}^{\mathsf{S}}_{j,n}[k_1,k_2,\dots,k_q] = 0, \tag{31}$$

uniformly in $n, k_1, k_2, \dots, k_q$.

Proof: It follows from Eqs. (9) and (30) that, when the wavelet packet functions are the functions $\psi^{\mathsf{S}}_{j,n}$, the cumulant $\mathsf{cum}^{\mathsf{S}}_{j,n}[k_1,k_2,\dots,k_q]$ of the discrete random process returned at node (j,n) by the Shannon M-DWPT of X satisfies

$$\begin{aligned}&|\mathsf{cum}^{\mathsf{S}}_{j,n}[k_1,k_2,\dots,k_q]| \\ &\qquad \leqslant \tfrac{M^{j(q+1)/2}\|\gamma_q\|_\infty}{(2\pi)^q}\int_{\Delta^q_{j,G(n)}} \mathrm{d}\omega_1\,\mathrm{d}\omega_2\dots\mathrm{d}\omega_q\end{aligned} \tag{32}$$

where $\Delta^q_{j,G(n)} = \underbrace{\Delta_{j,G(n)}\times\Delta_{j,G(n)}\times\dots\times\Delta_{j,G(n)}}_{q \text{ times}}$.

According to Eq. (10), $\displaystyle\int_{\Delta_{j,G(n)}} \mathrm{d}\omega = 2\pi/M^j$ so that $|\mathsf{cum}^{\mathsf{S}}_{j,n}[k_1,k_2,\dots,k_q]| \leqslant \|\gamma_q\|_\infty M^{-j(q-1)/2}$. Given any natural number $q>1$, the right hand side of the latter inequality does not depend on $n,k_1,\dots,k_N$ and vanishes when j tends to ∞, which completes the proof. ■

Henceforth, I_q stands for the $q\times q$ identity matrix and the zero-mean q-variate normal distribution with covariance matrix $\gamma(\omega_{\mathcal{P}})\mathrm{I}_q$ is denoted by $\mathcal{N}(0,\gamma(\omega_{\mathcal{P}})\mathrm{I}_q)$.

Corollary 1: With the same assumptions as those of theorems 1 and 3, let $\mathcal{P} = (\mathbf{U}^{\mathsf{S}}, \{\mathbf{W}^{\mathsf{S}}_{j,n_j}\}_{j\in\mathbb{N}})$ be a path of the Shannon M-DWPT tree for the decomposition of $\mathbf{U}^{\mathsf{S}}$. Let c^{S}_{j,n_j} stand for the discrete random process returned at node (j,n_j) by the projection of X on $\mathbf{W}^{\mathsf{S}}_{j,n_j}$.

Then, when j tend to infinity, the sequence $\left(c^{\mathsf{S}}_{j,n_j}\right)_j$ converges in the following distributional sense to a white Gaussian process with variance $\gamma(\omega_{\mathcal{P}})$: for every $x\in\mathbb{R}^q$ and every $\epsilon>0$, there exists a natural number $j_0 = j_0(x,\epsilon)$ such that, for every natural number $j \geqslant j_0$, the absolute value of the difference between the value at x of the probability distribution of the random vector

$$(c^{\mathsf{S}}_{j,n_j}[k_1], c^{\mathsf{S}}_{j,n_j}[k_2],\dots,c^{\mathsf{S}}_{j,n_j}[k_q])$$

and the value at x of the q-variate normal distribution $\mathcal{N}(0,\gamma(\omega_{\mathcal{P}})\mathrm{I}_q)$ is less than ϵ.

Proof: A straightforward consequence of theorems 1 and 3. ■

The following result describes the asymptotic behaviour of the cumulant of the discrete random process returned at node (j,n) in the case of practical interest where the M-DWPT of the PW space $\mathbf{U}^{\mathsf{S}}$ is achieved *via* decomposition filters satisfying Eq. (21).

Theorem 4: Let X be a zero-mean second-order strictly stationary random process, continuous in quadratic mean. Assume that the polyspectrum γ_q of X is an element of $L^\infty(\mathbb{R}^q)$ for every natural number $q\geqslant 1$ and that the spectrum γ of X is supported within $[-\pi,\pi]$. Consider the M-DWPT of the PW space $\mathbf{U}^{\mathsf{S}}$ when the decomposition filters $h^{[r]}_m$, $m=0,1,\dots,M-1$, satisfy Eq. (21).

Let $\mathsf{cum}^{[r]}_{j,n}$ stand for the cumulant of order $q+1$ of the discrete random process resulting from the projection of X on $\mathbf{W}^{[r]}_{j,n}$. We have, uniformly in $k_1,k_2,\dots,k_N$,

$$\lim_{r\to+\infty}\mathsf{cum}^{[r]}_{j,n}[k_1,k_2,\dots,k_N] = \mathsf{cum}^{\mathsf{S}}_{j,n}[k_1,k_2,\dots,k_N]. \tag{33}$$

Proof: By applying Eq. (30) to $\psi^{[r]}_{j,n}$ and $\psi^{\mathsf{S}}_{j,n}$, we obtain

$$\begin{aligned}&|\mathsf{cum}^{[r]}_{j,n}[k_1,k_2,\dots,k_q] - \mathsf{cum}^{\mathsf{S}}_{j,n}[k_1,k_2,\dots,k_q]| \\ &\quad\leqslant \frac{1}{(2\pi)^q}||\gamma_q||_\infty\int_{\mathbb{R}^q}\mathrm{d}\omega_1\dots\mathrm{d}\omega_q \\ &\qquad\Big|\mathcal{F}\psi^{[r]}_{j,n}(-\omega_1\dots-\omega_q)\mathcal{F}\psi^{[r]}_{j,n}(\omega_1)\dots\mathcal{F}\psi^{[r]}_{j,n}(\omega_q) \\ &\qquad -\mathcal{F}\psi^{\mathsf{S}}_{j,n}(-\omega_1\dots-\omega_q)\mathcal{F}\psi^{\mathsf{S}}_{j,n}(\omega_1)\dots\mathcal{F}\psi^{\mathsf{S}}_{j,n}(\omega_q)\Big|.\end{aligned} \tag{34}$$

The integrand on the right hand side of the second inequality above can now be upper-bounded by

$$2M^{j(q+1)/2}\mathcal{F}\Phi^{\mathsf{S}}(\omega_1)\mathcal{F}\Phi^{\mathsf{S}}(\omega_2)\dots\mathcal{F}\Phi^{\mathsf{S}}(\omega_q) \tag{35}$$

where we use Eqs. (23) and (24) and take into account that $|H^{[r]}_{m_\ell}(\omega)|$ and $|H^{\mathsf{S}}_{m_\ell}(\omega)|$ are less than or equal to 1. The upper-bound given by Eq. (35) is independent of r and integrable; its integral equals $2M^{j(q+1)/2}(2\pi)^q$. By taking Eq. (25) into account, we derive from Lebegue's dominated convergence theorem that the upper bound in Eq. (34) tends to 0 when r tends to $+\infty$. ■

Corollary 2: With the same assumptions as those of theorems 1 and 4, let $\mathcal{P} = (\mathbf{U}^{\mathsf{S}}, \{\mathbf{W}^{[r]}_{j,n_j}\}_{j\in\mathbb{N}})$ be a path of the M-DWPT tree of the PW space $\mathbf{U}^{\mathsf{S}}$, the decomposition being achieved by using filters $h^{[r]}_m$, $m=0,1,\dots,M-1$, satisfying Eq. (21). Let $c^{[r]}_{j,n_j}$ stand for the discrete random process returned at node (j,n_j) by the projection of X on $\mathbf{W}^{[r]}_{j,n_j}$.

Then, when j and r tend to infinity, the sequence $\left(c^{[r]}_{j,n_j}\right)_{r,j}$ converges in the following distributional sense to a white Gaussian process with variance $\gamma(\omega_{\mathcal{P}})$: for every $x\in\mathbb{R}^q$ and every $\epsilon>0$, there exists a natural number $j_0=j_0(x,\epsilon)$ such that, for every natural number $j\geqslant j_0$, there exists $r_0 = r_0(x,j,\epsilon)$ such that, for every order $r\geqslant r_0$, the absolute value of the difference between the value at x of the probability distribution of the random vector

$$(c^{[r]}_{j,n_j}[k_1], c^{[r]}_{j,n_j}[k_2],\dots,c^{[r]}_{j,n_j}[k_q])$$

and the value at x of the q-variate normal distribution $\mathcal{N}(0,\gamma(\omega_{\mathcal{P}})\mathrm{I}_q)$ is less than ϵ.

Proof: A consequence of theorems 1, 2, 3 and 4 that follows from Eqs. (20), (22), (31) and (33). ■

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